ON ANALYTIC FUNCTIONS OF CONSTANT MODULUS ON A GIVEN CONTOUR.*

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§1. Introduction.

Let A be a simply-connected area in the plane of the complex variable x, and C the limiting contour of A. We assume C to be composed of a finite number of parts of analytic curves, or more generally, to be such as to allow the conformal representation of A on the interior of a circle. Let $f(x)$ be an analytic function uniform in A and on C, and having only a finite number of essential singularities $c_1, c_2, \cdots, c_n$ inside A and on C. Finally let P be an infinite set of points on C, having at least one limit point distinct from any of the points $c_1, \cdots, c_n$ which may lie on C, and suppose that, K being a given constant,

$$|f(x)| = K \tag{1}$$

in all points of P. It is required to find the general analytic expression for $f(x)$.

As stated above, A may be represented conformally on the interior of a circle, so that it is sufficient to consider the case of A being the interior of the circle

$$|x| = R. \tag{2}$$

In §2, it will first be shown that under the given assumptions equation (1) holds not only for the points of P, but for all points on the circumference (2), except for such of the essential singular points as may lie on it. Furthermore, the principle of analytic continuation due to H. A. Schwarz will be applied to the determination of all essential singularities, poles and zeros of the function. In §3, the general expression for $f(x)$ is given in the simplest case, where there are no essential singularities inside or on the circumference, and finally §4 gives the desired analytic expression in the general case.

§2. Preliminary investigation of the properties of $f(x)$.

According to our assumption, there exists at least one limit point a of the point-set P for which $|f(x)| = K$, such that $|a| = R$ and is not coincident with any of the essential singularities of $f(x)$.

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Now the relation

$$x = a\frac{1 + iy}{1 - iy}, \qquad y = i\frac{a - x}{a + x}, \tag{3}$$

establishes a conformal representation of the interior of the circle (2) on the upper half plane, the circumference corresponding to the real axis, and $x = a$ to $y = 0$. In particular, the point-set P is transformed into a point-set Q on the real axis, one of the limit points of Q being $y = 0$.

Making

$$g(y) = f\left(a\frac{1 + iy}{1 - iy}\right) = f(x), \tag{4}$$

it follows from the assumptions made regarding $f(x)$, that $g(y)$ is uniform in the upper half plane and on the real axis, having only a finite number of essential singularities $d_1, d_2, \cdots, d_n$ $\left(\text{where } d_\nu = i\frac{a - c_\nu}{a + c_\nu} \neq 0\right)$ in the area considered and on its boundary, and furthermore $|g(y)| = K$ on the point-set Q.

If $y = 0$ were a pole of $g(y)$, we would have $|g(y)| > 2K$ in a certain vicinity of $y = 0$, which is contrary to our assumption that $y = 0$ is a limit point of Q. We therefore have, for a certain radius r,

$$g(y) = \sum_{\nu=0}^{\infty} (\alpha_\nu + i\beta_\nu)y^\nu, \qquad |y| < r,$$

where α_ν and β_ν are real quantities. Writing

$$\varphi(y) = \sum_{\nu=0}^{\infty} \alpha_\nu y^\nu, \qquad \psi(y) = \sum_{\nu=0}^{\infty} \beta_\nu y^\nu, \tag{5}$$

we have

$$g(y) = \varphi(y) + i\psi(y), \qquad |y| < r, \tag{6}$$

and the condition $|g(y)| = K$ for the point-set Q gives

$$\varphi(y)^2 + \psi(y)^2 = K^2 \tag{7}$$

for the points of Q verifying the condition $|y| < r$. The left hand member of (7) being holomorphic for $|y| < r$, and $y = 0$ being a limit point of Q, it follows that (7) holds identically for $|y| < r$. From (6) and (7) we obtain

$$\varphi(y) = \frac{1}{2}\left(g(y) + \frac{K^2}{g(y)}\right), \qquad \psi(y) = \frac{1}{2i}\left(g(y) - \frac{K^2}{g(y)}\right), \tag{8}$$

and from these equations, which define φ and ψ throughout the entire region of existence of $g(y)$, it is obvious that in the upper half plane and on the real axis, φ and ψ are uniform and have the essential singularities $d_1, \cdots, d_n$

only, and from (5) it follows that φ and ψ both assume real values on the part of the real axis given by $-r < y < r$.

Hence we may apply the principle of analytic continuation given by H. A. Schwarz,* and find that φ and ψ are uniform in the entire y-plane, having for essential singularities the points $d_1, \cdots, d_n$ and their conjugate points $d_1^0, \cdots, d_n^0$, and both assuming conjugate values for conjugate values of y. From the last property it follows that φ and ψ are both real in all points of the real axis (except for any of the essentially singular points which may lie on it), and consequently eq. (6) and (7), which now are proved valid in the entire y-plane, show *that* $|g(y)| = K$ *for all points of the real axis*, with the possible exception of some of the essential singularities. Hence $g(y)$ has neither zeros nor poles on the real axis. Equations (8) further show that a pole of either $\varphi(y)$ or $\psi(y)$ must be either a pole or a zero of $g(y)$ and conversely, that a pole or a zero of $g(y)$ must be a pole for both $\varphi(y)$ and $\psi(y)$. Thus the poles of $\varphi(y)$ are the same as those of $\psi(y)$ and coincide with the point-set formed by the zeros and poles of $g(y)$, and if $y = a$ belongs to this point-set, the same is the case with the conjugate point $y = a_0$. Let $y = a$ be a pole of $g(y)$ of the nth order; then a is a pole of the nth order of $\varphi(y)$ and $\psi(y)$, and these functions assuming conjugate values for conjugate values of y, $y = a_0$ must be a pole of the nth order of both $\varphi(y)$ and $\psi(y)$, that is, either a pole or a zero of the nth order of $g(y)$. Suppose a_0 to be a pole of $g(y)$; then we have the developments

$$g(y) = \frac{A}{(y-a)^n} + \cdots, \qquad g(y) = \frac{B}{(y-a_0)^n} + \cdots,$$

whence according to (8)

$$2\varphi(y) = \frac{A}{(y-a)^n} + \cdots, \qquad 2\varphi(y) = \frac{B}{(y-a_0)^n} + \cdots,$$

$$2\psi(y) = \frac{1}{i}\cdot\frac{A}{(y-a)^n} + \cdots, \qquad 2\psi(y) = \frac{1}{i}\cdot\frac{B}{(y-a_0)^n} + \cdots,$$

and as $\varphi(y)$ and $\varphi(y_0)$, $\psi(y)$ and $\psi(y_0)$ are conjugate quantities, we must have

$$B = A_0,$$

$$\frac{B}{i} = \left(\frac{A}{i}\right)_0 = -\frac{A_0}{i}, \qquad \text{or} \qquad B = -A_0,$$

whence $B = A_0 = A = 0$ contrary to our assumption. Therefore a_0 is a zero of the nth order of $g(y)$, and in the same way it is shown that if $y = a$ is a zero of the nth order of $g(y)$, $y = a_0$ is a pole of the nth order.

* H. A. Schwarz, "Über einige Abbildungsaufgaben," Journ. f. Math., *70* (1869), 105–120. See p. 107 of this paper or G. Darboux, Théorie des Surfaces, vol. I (1887), 174–176.

Returning to the variable x, it is known that to two conjugate values of y there correspond two points $x = a$ and $x = a'$, which are connected through the transformation by reciprocal radii

$$a' = \frac{R^2}{a_0}, \qquad a = \frac{R^2}{a_0'}, \tag{9}$$

and for $|x| = R$, we have

$$\left|\frac{x - a}{x - a'}\right| = \frac{|a|}{R}. \tag{10}$$

From what we have shown for the y-plane it then follows that $f(x)$ exists and is uniform in the entire x-plane; that two reciprocal points c_ν and c_ν' are both, or neither, essential singularities of $f(x)$, and that $x = a$ being a zero of the nth order (or $x = b$ a pole of the nth order) inside the circle $|x| = R$, $x = a'$ is a pole of the nth order (or $x = b'$ a zero of the nth order) outside the circle, and vice versa.

§3. The analytic expression for $f(x)$ when there are no essential singularities.

We now consider the particularly simple case where $f(x)$ has no essential singularities inside or on the circle $|x| = R$. According to the preceding paragraph, $f(x)$ then has no essential singularities outside of $|x| = R$, so that $f(x)$ is a rational function. Let

$$a_1,\ a_2,\ \cdots,\ a_m$$

be the zeros, and

$$b_1,\ b_2,\ \cdots,\ b_n$$

the poles of $f(x)$ inside $|x| = R$, each written as many times as indicated by its order; making

$$f_1(x) = \frac{\prod_{\nu=1}^{m} \dfrac{R(x - a_\nu)}{a_\nu(x - a_\nu')}}{\prod_{\nu=1}^{n} \dfrac{R(x - b_\nu)}{b_\nu(x - b_\nu')}} \tag{11}$$

(where, for $a_\nu = 0$, the factor $R(x - a_\nu)/a_\nu(x - a_\nu')$ has to be replaced by x/R, the same remark applying to any $b_\nu = 0$), and

$$f(x) = f_1(x) f_2(x), \tag{12}$$

it is seen that $f_2(x)$, being rational and having neither zeros nor poles, must be a constant, and as further, according to (10),

$$|f_1(x)| = 1 \qquad \text{for} \qquad |x| = R,$$

we have $|f_2(x)| = K$ for $|x| = R$ and consequently for every x, so that

$f_2(x) = Ke^{\gamma i}$, where γ is real. We thus finally obtain

$$f(x) = Ke^{\gamma i} \prod_{\nu=1}^{m} \frac{R(x - a_\nu)}{a_\nu(x - a_\nu')} \cdot \prod_{\nu=1}^{n} \frac{b_\nu(x - b_\nu')}{R(x - b_\nu)}. \tag{13}$$

This result is known* in the particular case where it is supposed from the beginning that $f(x)$ has a constant modulus on the entire circumference and not only in a point-set P. It may also be obtained in the following manner after showing that $|f(x)| = K$ on the entire circumference $|x| = R$, but without investigating the properties of $f(x)$ outside the circle. Forming (11) and (12) as previously, it is seen at once that $f_2(x)$ has neither zeros nor poles for $|x| \leqq R$, and that

$$|f_2(x)| = K \qquad \text{for} \qquad |x| = R,$$

whence

$$\left|\frac{1}{f_2(x)}\right| = \frac{1}{K} \qquad \text{for} \qquad |x| = R.$$

Both $f_2(x)$ and $1/f_2(x)$ being holomorphic for $|x| \leqq R$, it follows that

$$|f_2(x)| \leqq K \qquad \text{for} \qquad |x| \leqq R,$$

$$\left|\frac{1}{f_2(x)}\right| \leqq \frac{1}{K} \qquad \text{for} \qquad |x| \leqq R,$$

so that

$$|f_2(x)| = K \qquad \text{for} \qquad |x| \leqq R,$$

whence it is easily seen that $f_2(x) = \text{const.} = Ke^{\gamma i}$.

§4. The analytic expression for $f(x)$ in the general case.

We first remark that, for $|x| = R$, x and R^2/x are conjugate quantities, and consequently also

$$\frac{A}{(x - a)^n} \qquad \text{and} \qquad \frac{A_0}{\left(\frac{R^2}{x} - a_0\right)^n}.$$

By a simple transformation of the latter expression, we see that, for $|x| = R$,

$$\frac{A}{(x - a)^n} \qquad \text{and} \qquad \frac{(-1)^n A_0 a'^n x^n}{R^{2n}(x - a')^n} \tag{14}$$

are conjugate quantities.

Furthermore, let us distribute the zeros of $f(x)$ inside $|x| = R$ into n groups

* See, for instance, O. Blumenthal, "Sur le mode de croissance des fonctions entières," Bull. Soc. Math. de France, *35* (1907), 213–232, where it is obtained on pp. 214–215 by the aid of harmonic functions.

$$\begin{matrix} a_{11}, & a_{12}, & \cdots, & a_{1\lambda}, & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}, & a_{n2}, & \cdots, & a_{n\lambda}, & \cdots \end{matrix}$$

such that

$$\lim_{\lambda=\infty} a_{\nu\lambda} = c_\nu \qquad (\nu = 1, 2, \cdots, n),$$

and similarly the poles inside $|x| = R$:

$$\begin{matrix} b_{11}, & b_{12}, & \cdots, & b_{1\lambda}, & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1}, & b_{n2}, & \cdots, & b_{n\lambda}, & \cdots \end{matrix}$$

so that

$$\lim_{\lambda=\infty} b_{\nu\lambda} = c_\nu \qquad (\nu = 1,\ 2,\ \cdots\ n)\ *$$

We now form the primary factor

$$E(x; a_{\nu\lambda}, a_{\nu\lambda}'; c_\nu, c_\nu') = \frac{1 - \dfrac{a_{\nu\lambda} - c_\nu}{a_{\nu\lambda}} \cdot \dfrac{x}{x - c_\nu}}{1 - \dfrac{a_{\nu\lambda}' - c_\nu'}{x - c_\nu'}} \cdot e^{\sum_{\mu=1}^{n_\lambda - 1} \frac{1}{\mu} \left[\left(\frac{a_{\nu\lambda} - c_\nu}{a_{\nu\lambda}} \frac{x}{x - c_\nu} \right)^\mu - \left(\frac{a_{\nu\lambda}' - c_\nu'}{x - c_\nu'} \right)^\mu \right]},$$

where n_λ is an integer to be determined later.

For $|x| = R$, we have by (10)

$$\left| \frac{1 - \dfrac{a_{\nu\lambda} - c_\nu}{a_{\nu\lambda}} \dfrac{x}{x - c_\nu}}{1 - \dfrac{a_{\nu\lambda}' - c_\nu'}{x - c_\nu'}} \right| = \left| \frac{c_\nu}{a_{\nu\lambda}} \frac{x - c_\nu'}{x - c_\nu} \frac{x - a_{\nu\lambda}}{x - a_{\nu\lambda}'} \right| = \left| \frac{c_\nu}{a_{\nu\lambda}} \right| \cdot \frac{R}{|c_\nu|} \cdot \frac{|a_{\nu\lambda}|}{R} = 1,$$

and according to (14), the two terms inside the bracket in the exponential are conjugate quantities for $|x| = R$. Therefore we have

(16) $$|E(x; a_{\nu\lambda}, a_{\nu\lambda}'; c_\nu, c_\nu')| = 1 \qquad \text{for} \qquad |x| = R.$$

In the case $c_\nu = 0$, $c_\nu' = \infty$, the expression (15) should be replaced by

(15a) $$E(x; a_{\nu\lambda}, a_{\nu\lambda}'; 0, \infty) = \frac{1 - \dfrac{a_{\nu\lambda}}{x}}{1 - \dfrac{x}{a_{\nu\lambda}'}} e^{\sum_{\mu=1}^{n_\lambda - 1} \frac{1}{\mu} \left[\left(\frac{a_{\nu\lambda}}{x} \right)^\mu - \left(\frac{x}{a_{\nu\lambda}'} \right)^\mu \right]},$$

where we obviously have

(16a) $$|E(x; a_{\nu\lambda}, a_{\nu\lambda}'; 0, \infty)| = 1 \qquad \text{for} \qquad |x| = R,$$

and in the case of $a_{\nu\lambda} = 0$, $a_{\nu\lambda}' = \infty$ we write

(15b) $$E(x; 0, \infty; c_\nu, c_\nu') = \frac{x}{R},$$

* In the case where c_ν is not a limit point of poles or zeros, the corresponding primary factors should be suppressed in equations (15) to (17).

so that

$$|E(x;\ 0,\ \infty;\ c_\nu,\ c_\nu')| = 1 \qquad \text{for} \qquad |x| = R. \tag{16b}$$

It is shown in the classical way,* that by conveniently determining n_λ (for instance, $n_\lambda = \lambda$), the infinite product

$$\prod_{\lambda=1}^{\infty} E(x;\ a_{\nu\lambda},\ a_{\nu\lambda}';\ c_\nu,\ c_\nu')$$

is uniformly convergent in any part of the plane for which c_ν, c_ν' and $a_{\nu 1}'$, $a_{\nu 2}'$, $\cdots$ are exterior points, and that the product represents a uniform analytic function of x, having c_ν and c_ν' for its essential singularities, $a_{\nu 1}$, $a_{\nu 2}$, $\cdots$ for its zeros, $a_{\nu 1}'$, $a_{\nu 2}'$, $\cdots$ for its poles, and finally of modulus $= 1$ for $|x| = R$.

Now make

$$f(x) = \prod_{\nu=1}^{n} \prod_{\lambda=1}^{\infty} E(x;\ a_{\nu\lambda},\ a'_{\nu\lambda};\ c_\nu,\ c_\nu') \cdot f_1(x); \tag{17}$$

then $f_1(x)$ has neither zeros, nor poles, nor essential singularities outside of the points c_ν, c_ν' $(\nu = 1, 2, \cdots, n)$, and furthermore

$$|f_1(x)| = K \quad \text{for} \quad |x| = R.$$

Let $c_1, c_2, \cdots, c_m$ be those of the points $c_1, \cdots, c_n$ that lie inside, and $c_{m+1}, \cdots, c_n$ those that lie on the circumference $|x| = R$; as obviously $c_\nu' = c_\nu$ for $\nu = m + 1, \cdots, n$, the only possible zeros or singularities of $f_1(x)$ are $c_1, \cdots, c_m;\ c_1', \cdots, c_m';\ c_{m+1}, \cdots, c_n$.

In the vicinity of $x = c_\nu$ $(\nu = 1, 2, \cdots, n)$ we have the following development (Weierstrass, l. c.):

$$f_1(x) = (x - c_\nu)^{l_\nu} e^{G_\nu\left(\frac{1}{x-c_\nu}\right) + P_\nu(x-c_\nu)}, \tag{19}$$

where l_ν is an integer and

$$G_\nu\left(\frac{1}{x - c_\nu}\right) = \sum_{\lambda=1}^{\infty} A_{\nu\lambda}\left(\frac{1}{x - c_\nu}\right)^\lambda \tag{20}$$

an integral function of its argument. Writing

$$\varphi_\nu(x) = \left(\frac{R}{c_\nu}\,\frac{x - c_\nu}{x - c_\nu'}\right)^{l_\nu} e^{\sum\limits_{\lambda=1}^{\infty}\left[\frac{A_{\nu\lambda}}{(x-c_\nu)^\lambda} + \frac{(-1)^{\lambda+1}(A_{\nu\lambda})_0 c_\nu'^{\lambda} x^\lambda}{R^{2\lambda}(x-c_\nu')^\lambda}\right]}, \tag{21}$$

* See, for instance, K. Weierstrass, "Zur Theorie der eindeutigen analytischen Funktionen," Abhandl. d. Ak. d. Wiss. Berlin, 1876, 11–60, or Abhandlungen aus der Funktionenlehre, 1–52, or Math. Werke, II, 77–124; G. Mittag-Leffler, "Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante," Acta Math., *4* (1884), 1–79.

or, respectively, for $c_\nu = 0$,

$$\varphi_\nu(x) = \left(\frac{x}{R}\right)^{l_\nu} e^{\sum_{\lambda=1}^{\infty}\left[\frac{A_{\nu\lambda}}{x^\lambda} - \frac{(A_{\nu\lambda})_0 x^\lambda}{R^{2\lambda}}\right]}, \tag{21a}$$

$\varphi_\nu(x)$ has no zeros or singularities except c_ν, c_ν', and by (10) and (14)

$$|\varphi_\nu(x)| = 1 \quad \text{for} \quad |x| = R. \tag{22}$$

Now let us make

$$f_1(x) = \varphi_1(x)\varphi_2(x)\cdots\varphi_m(x)\cdot f_2(x); \tag{23}$$

then, by (19) and (21), $f_2(x)$ is holomorphic and different from zero for $x = c_\nu$ ($\nu = 1, 2, \cdots, m$), and by (18) and (22),

$$|f_2(x)| = K \quad \text{for} \quad |x| = R. \tag{24}$$

Accordingly, $f_2(x)$ is also holomorphic and different from zero for $x = c_\nu'$ ($\nu = 1, 2, \cdots, m$), so that the only possible singularities or zeros of $f_2(x)$ are $x = c_{m+1}, \cdots, c_n$, and in the vicinity of each of these we have, by (19) and (23),

$$f_2(x) = (x - c_\nu)^{l_\nu} e^{G_\nu\left(\frac{1}{x-c_\nu}\right) + P_{\nu 1}(x-c_\nu)} \qquad (\nu = m+1, \cdots, n). \tag{25}$$

Now make the conformal representation

$$y = i\frac{c_\nu - x}{c_\nu + x};$$

then in the vicinity of $y = 0$ (corresponding to $x = c$) we have

$$\log f_2(x) = l_\nu \log\left(\frac{-2c_\nu y}{y+i}\right) + \sum_{\lambda=1}^{\infty}\frac{B_{\nu\lambda} + iC_{\nu\lambda}}{y^\lambda} + \sum_{\lambda=0}^{\infty}(\bar{B}_{\nu\lambda} + i\bar{C}_{\nu\lambda})y^\lambda,$$

where all B and C are real quantities. For y real (corresponding to $|x| = R$) the real part of this expression is

$$\tfrac{1}{2}l_\nu \log\frac{y^2}{y^2+1} + l_\nu \log(2R) + \sum_{\lambda=1}^{\infty}\frac{B_{\nu\lambda}}{y^\lambda} + \sum_{\lambda=0}^{\infty}\bar{B}_{\nu\lambda}y^\lambda,$$

and as, according to (24), this is equal to $\log K$, we must have

$$l_\nu = 0, \qquad B_{\nu\lambda} = \bar{B}_{\nu\lambda} = 0 \quad \begin{pmatrix}\lambda = 1, 2, 3, \cdots \\ \nu = m+1, \cdots, n\end{pmatrix}.$$

Consequently, making

$$\varphi_\nu(x) = e^{\sum_{\lambda=1}^{\infty} iC_{\nu\lambda}\left(\frac{c_\nu+x}{i(c_\nu-x)}\right)^\lambda} = e^{\sum_{\lambda=1}^{\infty} i^{\lambda+1}C_{\nu\lambda}\left(\frac{x+c_\nu}{x-c_\nu}\right)^\lambda} \qquad (\nu = m+1, \cdots, n), \tag{26}$$

φ_ν has neither zeros nor poles, its only possible essential singularity is c_ν, and

$$|\varphi_\nu(x)| = 1 \quad \text{for} \quad |x| = R. \tag{27}$$

Now writing

$$f_2(x) = \varphi_{m+1}(x) \cdots \varphi_n(x) \cdot f_3(x), \tag{28}$$

$f_3(x)$ has no singularities and therefore is a constant; furthermore $|f_3(x)| = K$ for $|x| = R$, so that finally

$$f_3(x) = Ke^{\gamma i}. \tag{29}$$

We have thus solved the proposed problem, and it is evident that the method used may be extended to the case where the essential singularities of $f(x)$ inside or on $|x| = R$, instead of being finite in number, form a point-set belonging to the general class considered by Mittag-Leffler in the paper previously quoted.